MA 161: Lesson 36

Today: Exponential growth and decay (7.2)

Announcements:

* No Class on Friday - evening exam compensation.
* No Office Hours on Friday
* Final exam on Monday (12/15/25): 1pm - 3pm
* Review for Final - Next week

Office Hours: MWF 2:45PM - 4:15PM

Warmup!

* $5 = e^{3t} \rightsquigarrow$ find value of t

$\ln 5 = \ln(e^{3t}) = 3t \ln e = 3t \rightsquigarrow t = \frac{\ln 5}{3}$.

* Find a function $y(t)$ such that $y'(t) = 5y(t)$

$\frac{d}{dt} e^t = e^t$

$\frac{d}{dt} e^{5t} = 5 \cdot e^{5t}$

Exponential Growth / Decay

$y(t)$ = Size of a Quantity at a given time t

$\underbrace{\text{Rate of change}}_{\frac{dy}{dt}}$ is proportional to $\underbrace{\text{its size}}_{y(t)}$

$$\boxed{\frac{dy}{dt} = K\,y(t)}$$

$K > 0 \rightsquigarrow$ Growth

$K < 0 \rightsquigarrow$ decay

K = proportionality Constant

OR

Relative Rate of change

Solving $\frac{dy}{dt} = ky$

eg: $k=1 \rightsquigarrow \left.\frac{dy}{dt} = y\right\}$ function whose derivative is itself

$y(t) = c\,e^t$

$y(0) = c \cdot e^0 \rightsquigarrow \left. c = y(0)\right\}$ initial value.

$$\left.\underbrace{y(t) = y_0 e^{kt}}\right\} \text{ is solution to } \frac{dy}{dt} = ky, \quad y(0) = y_0$$

$$\frac{dy}{dt} = y_0 \cdot \frac{d}{dt} e^{kt}$$

$$= y_0 k \cdot e^{kt} = k \cdot y_0 e^{kt}$$

$$= k \cdot y(t)$$

$$y(0) = y_0 \cdot e^0 = y_0.$$

The population of a town is decreasing at a relative rate of 5% per year. In year 2005, the population is 15000. What is the population t years after 2005.

$k<0$

$5\% = \frac{5}{100} = 0.05$

$k = -0.05$

2005 ⇝ population = 15000 ⇝ $y(0) = y_0 = 15000$.

Assume $t=0$ in year 2005

exp. Model: $y(t) = y_0 e^{kt}$

$y(t) = 15000\, e^{-0.05t}$

What is population in year 2025 ⇝ $t=20$

$y(20) = 15000\, e^{-0.05*20} = 15000 * e^{-1} \approx 5500$

in What year would population Reduce to 3000.

$$y(t) = 15000\, e^{-0.05t}$$

find t when $y(t) = 3000$

$$3000 = 15000\, e^{-0.05t}$$

$$5^{-1} = e^{-0.05t}$$

$$\ln 5^{-1} = \ln e^{-0.05t} \Bigg\} \quad -\ln 5 = -0.05\, t$$

$$\Rightarrow t = \frac{\ln 5}{0.05} = 20 \ln 5$$

$$\approx 33$$

$$\text{Year} = 2005 + 20 \ln 5 \approx 2038$$

population decreases to 3000.

Relative Growth/Decay Rate v/s Average Growth/Decay Rate

Relative Growth/Decay Rate → gives k value directly

eg: decreases at a relative rate of 5%

$k = -0.05$

Average Growth/Decay Rate: decreases at an "ave" rate of 5%.

$y(1) = 5\%$ Less than $y(0)$

$= 0.95 y_0$

use this to compute k value

$0.95 y_0 = y_0 e^{k \cdot 1}$

$k = \ln(0.95)$

$\neq -0.05$

If a donut costs $1 in 2016, assuming the relative inflation rate stays constant at 3%

1) what will a donut cost in 2030?

2) when will the price of donut be 3$

$k > 0$

$k = \frac{3}{100} = 0.03$

2016 $\rightsquigarrow$ $t=0$ $\rightsquigarrow$ $y(0) = 1\$$.

exp. Model!! $y(t) = y_0 e^{kt}$

$$\boxed{y(t) = e^{0.03t} \$}$$

1) Year 2030 $\rightsquigarrow$ $t = 14$, $\text{cost} = y(14) = e^{0.03 * 14} \$ = e^{0.42} \$ \approx 1.52\$$.

2) Find t when $y(t) = 3\$$

$3 = e^{0.03t} \Rightarrow \ln 3 = 0.03t \Rightarrow t = \frac{\ln 3}{0.03} \approx 38$.

$\rightarrow$ Year $= 2016 + 38 = 2054$.

The half-life of caffeine in human body is 6 hours. If 100mg of caffeine is ingested by drinking coffee at 3pm.
1)How much caffeine remains t hours after 3pm?
2)How much caffeine remains at 11pm?
2)At what time there will be 20mg caffeine left in the body?

in 6 hours, the amount Reduces to half

Suppose $y(0) = y_0$, $y(6) = \frac{y_0}{2}$

$y(t) = y_0 e^{kt} \rightsquigarrow$ $y(6) = y_0 e^{k \cdot 6}$

$\frac{y_0}{2} = y_0 \cdot e^{6k} \rightsquigarrow \frac{1}{2} = e^{6k} \rightsquigarrow -\ln 2 = 6k$

$k = -\frac{\ln 2}{6} < 0,$

3pm $\rightsquigarrow$ $t = 0$, $y_0 = 100$

1) t hours after 3pm

$$\boxed{y(t) = 100\, e^{-\frac{\ln 2}{6} t}}$$

$$y(t) = 100\, e^{-\frac{\ln 2}{6} t}$$

2) caffeine at 11pm $\rightsquigarrow$ 8 hours after 3pm
$= y(8)$

$$y(8) = 100\, e^{-\frac{\ln 2}{6} * 8} = 100 \cdot e^{-\frac{4}{3}\ln 2}$$

$$= 100 * 2^{-4/3} \approx \boxed{40mg}$$

3) $20 \text{ mg} = \frac{1}{2}\, 40mg$ $\rightsquigarrow$ 6 hours from 11pm to reduce to $\frac{1}{2}$ of 40mg.

half life

$\rightsquigarrow$ 5AM.

$$20 = 100\, e^{-\frac{\ln 2}{6} * t} \rightsquigarrow 5^{-1} = e^{-\frac{\ln 2}{6} t} \Rightarrow \ln 5 = \frac{\ln 2}{6} t \Rightarrow t = \frac{6 \ln 5}{\ln 2} \approx 14$$

14 hours After 3pm $\rightsquigarrow$ 5AM.

The half-life of C-14 is 5730 years used to date objects. Suppose an object has 30% of its C-14 remaining. How old is the object?

Suppose $y(0) = y_0$

$y(5730) = \frac{y_0}{2}$

$y(t) = y_0 e^{kt}$

$y(5730) = y_0 e^{k*5730}$

$\frac{\cancel{y_0}}{2} = \cancel{y_0} e^{k*5730}$

$-\ln 2 = k * 5730 \rightsquigarrow \boxed{k = \frac{-\ln 2}{5730}}$

Find t when $y(t) = 30\%$ of y_0

$= 0.3 y_0$

$0.3 \cancel{y_0} = \cancel{y_0} e^{\frac{-\ln 2}{5730} t} \rightsquigarrow \ln(0.3) = \frac{-\ln 2}{5730} t$

$t = \frac{5730 * \ln(0.3)}{-\ln(2)} \approx 10{,}000$ yrs.

half life = 5730

What is Age when 25% of object Remain

$$\begin{tikzcd} y_0 \arrow[r, "5730"] \arrow[rr, bend right, "2 \times 5730 = 11460"'] & \frac{y_0}{2} \arrow[r, "5730"] & \underbrace{\frac{1}{2}\left(\frac{y_0}{2}\right) = \frac{y_0}{4}}_{25\% \text{ of } y_0} \end{tikzcd}$$